DIFFERENTIAL EQUATIONS DESCRIBING THE CHANGES OF SHAPE CAUSED BY KARST CORROSION OF ANY ARBITRARY LIMESTONE SURFACE

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Abstract: A sort of karstic processes happens in circumstances when the rock surface exposed to solution is identical with the surface that determines the macroscopic shape of the rock, i.e. the solution takes place directly on the surface of the rock that is not covered by soil or debris. Such processes occur at the making of karrenfelds, rillenkarren, rock-bowls as well as at the making of vertical elements (potholes, vertical shafts). Though these forms seem to be substantially different, their development can be described with a universal differential equation. This paper describes the setting up of the differential equation system.

Introduction

A sort of karstic processes happens in circumstances when the rock surface exposed to solution is identical with the surface that determines the macroscopic shape of the rock, i.e. the solution takes place directly on the surface of the rock that is not covered by soil or debris. Such processes occur at the making of karrenfelds, rillenkarren, rock-bowls as well as at the making of vertical elements (potholes, vertical shafts). Though these forms seem to be substantially different, their development can be described with a universal differential equation.

This paper describes the setting up of the differential equation system. The equation system (due to its universal validity) is relatively complicated but in concrete cases it reduces naturally if the symmetries and other reducing conditions of the studied case are made use of.

The setting up of such a differential equation system is justified because it unifies the efforts (virtually very different from one another regarding the different nature of their specific aims) of others having studied the dissolution of the free limestone surface *(VERESS, M.—PÉNTEK, K. 1990, 1992, DUBLJANSZKIJ, J. V. 1989, JAKUCS, L. 1971, SZUNYOGH, G. 1995 etc.)* and the method offers new theoretical views on other phenomena that have not been yet studied. The basic novelty of the method is that it offers possibilities for the theoretical modeling of various karst-forms, their origin and the explanation of their morphological properties.

The Model of the Karstification of Uncovered Limestone Surfaces

A q_V volume of rainfall falls on the limestone surface shown on *Fig. 1*. q_V determines the direction of the rainfall and the volume of water falling at right angle at a unit surface at unit time. As a result water will flow on the rock surface with good approximation along the slope lines of the surface. (Only an approximation because if the horizontal projection of the slope line is a curve, the centrifugal force will drive the water particles from the slope lines, they go astray in the bends.) The volume of water flowing on the surface is not originated solely in the rain, but it is boosted by volumes coming from other parts of the surface outside of the studied area.

As the atmosphere contains 0.03% of carbon dioxide, rainwater becomes slightly acidic and dissolves limestone. While the rainwater flows on the rock surface its aggressive nature decreases due to the solution of limestone and also increases because of the addition of new rainwater still rich in carbonic acid content. As a result the aggressivity of the water-film will be different at the various points of the rock surface. For the denudation (corrosion) is the faster if the water is the more aggressive, the shape of the rock surface will be varied in space and time.
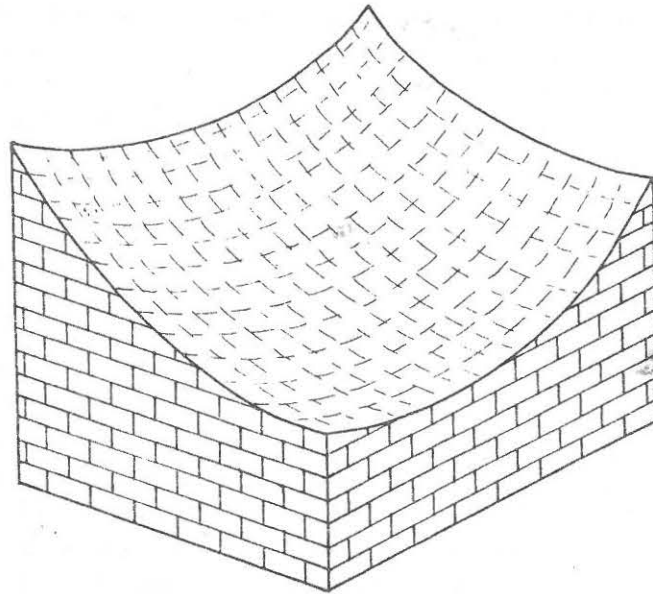
Figure 1. Detail of limestone surface exposed to solution

The Determination of the Differential Equation of the Function that Describes the Shape of the Rock Surface

At the modeling of the exposed limestone surface *the listed actions will be taken to consideration:*

1) The volume of calcium carbonate departing from the limestone in a time unit (the rate of dissolution) will be directly proportional with the aggressivity of the water flowing above it, that is: with the difference of the de facto calcium carbonate concentration and that of the saturated solution.

2) The calcium carbonate concentration of the water-film flowing on the surface will be increased by the dissolution of limestone but it is decreased by the increase of the solvent (rainfall on the surface).

3) The shift per time unit of an arbitrary point of the limestone surface (denudation) is directly proportional with the volume of rock that departs from that point in per time unit. With other words: where corrosion is quicker, quicker is the surface denudation.

4) The direction of the flow velocity of the liquid-film (at the definite point) is determined by the slope direction of the surface, its magnitude by the thickness and discharge of the liquid-film.

5) The thickness of the liquid-film (as for the laws of flow of free-surface viscous liquids) is basically determined by the volume of discharge and the gradient of the water tracks.

In the computation the listed quantities will be taken as well defined quantities

— the velocity of the flow from neighboring areas at the fringe of the studied area;

— the direction and volume of the rainfall;

— parameters defining the velocity of limestone dissolution;

— the saturation concentration of calcium carbonate of the water;

— elemental hydraulic and petrologic parameters.

In the computation the listed quantities will be taken as undefined quantities

— the formula that defines the limestone surface;

— the direction and measure of the velocity of water flow at an arbitrary point of the limestone surface;

— the concentration of calcium carbonate dissolved in water at any point of the limestone surface;

— the thickness of the fluid-film along the surface;

— the direction and degree of the limestone slope (related to space and time).

All undefined quantities shall be regarded as functions of the three spatial coordinates and time.

The Means of Mathematical Definition of the Limestone Surface

The formula that defines the limestone surface — to avoid the ruining of generality — will be sought in a coordinate-free form (that is: invariable regarding the types and position of coordinate systems) to enable the use of the most applicable coordinate system for the solution of the specific task. *(SZUNYOGH, G. 1994)* The vector pointing to an arbitrary point of the surface will be **r** and the equation to determine the surface is

$$t = F(\mathbf{r}) \tag{1}$$

where t is time and $F(\mathbf{r})$ is to be understood as a vectorial-scalar function. E.g. in the case of Descartian coordinates

$$t = F(x, y, z). \tag{2}$$

The (1) explicitly determines the time when the surface "passes" the point marked by the **r** vector. Implicitly it shows that at the t time what x, y, z coordinates are on the surface. If the $F(x,y,z)$ becomes known (as the solution of its differential equation) than figuring z from (2) z can be expressed explicitly as a function of the x and y spatial coordinates and t time.

$$z = G(x, y, t). \tag{3}$$

The expression of the limestone surface is better as in (1) than in (3) because it does not emphasize either spatial coordinate "suppressing" the others as does (3) with z when it may happen that in certain cases x or y would be the variants more fitting to the solution.

(1) is suitable to the mathematical description in the

$$t = F(r, \varphi, z) \tag{4}$$

form using cylindrical or

$$t = F(R, \varphi, \vartheta) \tag{5}$$

form using spherical coordinates.

The Velocity of the Shift of the Limestone Surface

Let's regard the position of a t point and it's position in a commencing $t + \Delta t$ time *(Fig. 2)*
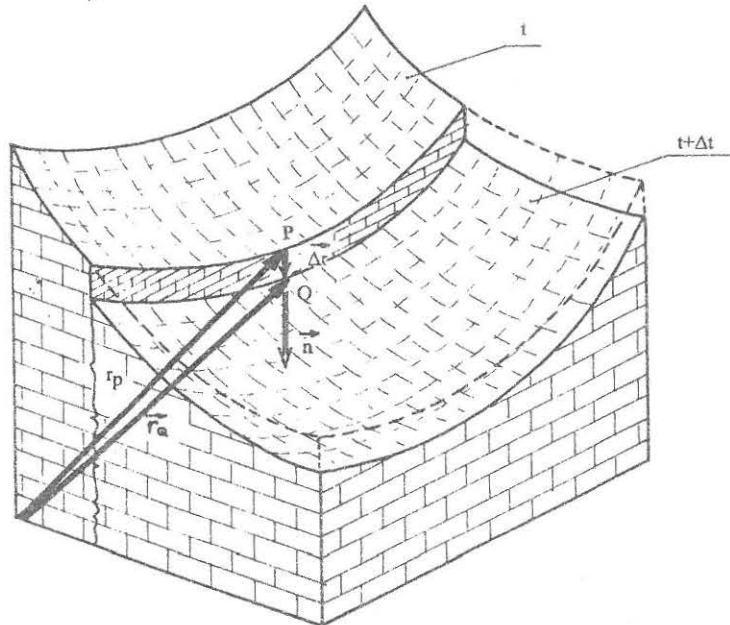

Figure 2. The shifting of the points of the rock surface during its dissolution

A P point will be selected on the surface belonging to t time and a Q pint belonging to the $t + \Delta t$ time doing it in a way that the PQ section shall be at right angle to the tangent plane of the $\Delta \mathbf{r} = \Delta r \cdot \mathbf{n}$ surface at P (or the PQ section shall be identical with the normal of the surface). It should point at the $\mathbf{r}_P$ P, the $\mathbf{r}_Q$ Q point. The $\Delta \mathbf{r}$ vector pointing from P to Q

$$\Delta \mathbf{r} = \mathbf{r}_Q - \mathbf{r}_P. \tag{6}$$

Consequently:

$$\Delta \mathbf{r} = \Delta r \cdot \mathbf{n}, \tag{7}$$

where Δr stands for the absolute value of the $\Delta \mathbf{r}$ vector. The velocity of the shifting of the surface will be:

$$\mathbf{w} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}. \tag{8}$$

The $\mathbf{w}$ shall be expressed with its $F(\mathbf{r})$ gradient. To achieve this (8) shall be reordered as follows:

$$\mathbf{w} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} \cdot \mathbf{n} = \lim_{\Delta t \to 0} \frac{1}{\dfrac{\Delta t}{\Delta r}} \mathbf{n}. \tag{9}$$

As Δt converges zero point Q converges P beyond limits, so the $\Delta \mathbf{r} = \mathbf{r}_Q - \mathbf{r}_P$ vector converges zero itself. So it can be expressed:

$$\mathbf{w} = \lim_{\Delta r \to 0} \frac{1}{\dfrac{\Delta t}{\Delta r}} \mathbf{n}. \tag{10}$$

According to the rules of limit determination:

$$\lim_{\Delta r \to 0} \frac{1}{\dfrac{\Delta t}{\Delta r}} = \frac{1}{\lim\limits_{\Delta r \to 0} \dfrac{\Delta t}{\Delta r}}, \tag{11}$$

(unless the denominator of the right side is not zero) thus:

$$\mathbf{w} = \frac{1}{\lim\limits_{\Delta r \to 0} \dfrac{\Delta t}{\Delta r}} \mathbf{n}. \tag{12}$$

Δt can be expressed with the help $F(\mathbf{r})$ formula. As point P is situated on the surface belonging to t time:

$$F(\mathbf{r}_P) = t, \tag{13}$$

point Q is situated on the surface described by $t+Dt$, thus

$$F(\mathbf{r}_Q) = t + \Delta t. \tag{14}$$

The difference of (14) and (13)

$$\Delta t = F(\mathbf{r}_Q) - F(\mathbf{r}_P). \tag{15}$$

Using the equations (6) and (7)

$$\Delta t = F(\mathbf{r}_P + \Delta r \cdot \mathbf{n}) - F(\mathbf{r}_P). \tag{16}$$

Inserting Δt to the denominator of (12)

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$$w = \cfrac{1}{\lim\limits_{\Delta r \to 0} \cfrac{F(\mathbf{r}_P + \Delta r \cdot \mathbf{n}) - F(\mathbf{r}_P)}{\Delta r}} \mathbf{n} \qquad (17)$$

is resulted. It can be recognized that the expression in the denominator of (12) is nothing else but the derivative of $F(\mathbf{r})$ in the $\mathbf{n}$ direction

$$\lim\limits_{\Delta r \to 0} \frac{F(\mathbf{r}_P + \Delta r \cdot \mathbf{n}) - F(\mathbf{r}_P)}{\Delta r} = \frac{\partial F}{\partial n}. \qquad (18)$$

It is well known that directional derivatives can be expressed by the gradients of the function to be derivated, so

$$\frac{\partial F}{\partial n} = \mathbf{n} \cdot \operatorname{grad} F. \qquad (19)$$

As the $\mathbf{n}$ normal of the surface is identical with the direction of the gradient:

$$\mathbf{n} = \frac{\operatorname{grad} F}{\left| \operatorname{grad} F \right|}. \qquad (20)$$

Consequently

$$\frac{\partial F}{\partial n} = \operatorname{grad} F \cdot \frac{\operatorname{grad} F}{\left| \operatorname{grad} F \right|} = \frac{\operatorname{grad} F \cdot \operatorname{grad} F}{\left| \operatorname{grad} F \right|} = \frac{\left(\operatorname{grad} F \right)^2}{\left| \operatorname{grad} F \right|}. \qquad (21)$$

Substituting (21) to (17)

$$w = \frac{\left| \operatorname{grad} F \right|}{\left(\operatorname{grad} F \right)^2} \mathbf{n}. \qquad (22)$$

substituting (20) to the place of $\mathbf{n}$ and reducing it the next formula can be achieved for the w velocity of the shift of the rock surface.

$$w = \frac{\operatorname{grad} F}{\left(\operatorname{grad} F \right)^2} \qquad (23)$$

The Density of the Material Flux of Limestone into the Solution

The density of the material flux of limestone is the quantity of limestone removed from a unit area of the limestone surface in unit time

$$q_K = \lim_{\Delta t \to 0, \, \Delta A \to 0} \frac{\Delta m_K}{\Delta t \cdot \Delta A}, \tag{24}$$

where Δm_K is the quantity of calcium carbonate dissolved in some Δt time from a ΔA surface area.

For the determination of Δm_K a minor ΔA size area of the continuously denudating rock surface at t time shall be chosen.

As the surface of the rock is dissolved at a — according to the earlier coding — **w** velocity, the denudation will be in a Δt time interval:

$$\Delta r = \mathbf{w} \cdot \mathbf{n} \cdot \Delta t \tag{25}$$
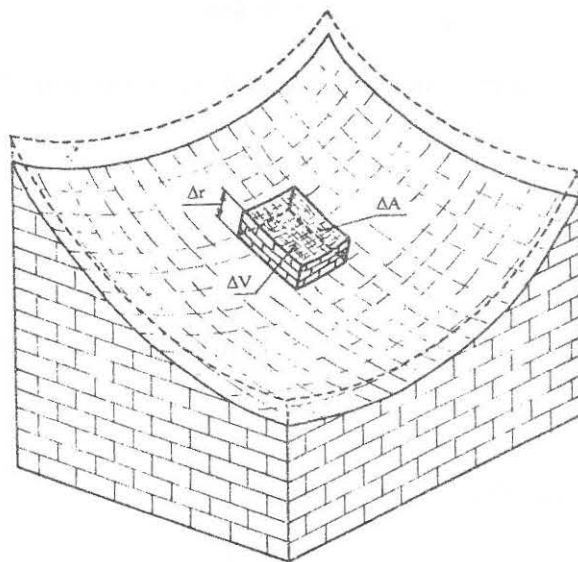

Figure 3. The presentation of the volume originally occupied by limestone dissolved from the ΔA surface element during Δt time

If ΔA is minute enough (that is $\Delta A \to 0$), than the surface of the rock can be approximated by a plane within ΔA-n and the value of **w** can be considered constant. Consequently the original volume of the limestone removed in the Δt time interval (see *Fig. 3*):

$$\Delta V = \Delta A \cdot \Delta r, \tag{26}$$

or the mass of the removed rock:

$$\Delta m_K = \rho_K \cdot \Delta V, \tag{27}$$

where ρ_K is the density of the limestone.

Considering (24), (25) and (26)

$$\Delta m_K = \rho_K \cdot \Delta A \cdot \mathbf{w} \cdot \mathbf{n} \cdot \Delta t, \tag{28}$$

this written into (24) the density of the material flux of calcium carbonate removed from the limestone during solution is received

$$q_K = \lim_{\Delta t \to 0,\ \Delta A \to 0} \frac{\rho_K \cdot \Delta A \cdot \mathbf{w} \cdot \mathbf{n} \cdot \Delta t}{\Delta t \cdot \Delta A}. \tag{29}$$

Formula (29) after some reduction and limit determination can be given in this form:

$$q_K = \mathbf{w} \cdot \mathbf{n} \tag{30}$$

Considering the form of $\mathbf{w}$ determined by (22) the scalar product indicated in (30) will be put into the form:

$$\mathbf{w} \cdot \mathbf{n} = \frac{|\operatorname{grad} F|}{\left(\operatorname{grad} F\right)^2} \mathbf{n} \cdot \mathbf{n} \tag{31}$$

Using that $\mathbf{n} \cdot \mathbf{n} = 1$, the density of the material flux of limestone will be:

$$q_K = \frac{\rho_K}{|\operatorname{grad} F|} \tag{32}$$

The Calcium Carbonate Concentration of the Water Flowing on the Rock Surface

The calcium carbonate concentration of the fluid-film should be determined by such a function that depends only on two coordinates (e.g. x and y) as the flow itself varies on a two dimensional surface. For example in a "normally positioned" Descartian coordinate system the general form of this function would be

$$c = c(x, y, t), \tag{33}$$

that expresses that on a point of x, y coordinates on the surface of the limestone body in question the calcium carbonate concentration is exactly $c(x,y,t)$.

It is proposed to deviate however from this type of writing using the fact that there is an unambiguous relation between the z elevation and t time according to (3), so (avoiding the ruining of generality) it can be determined

in the function of the three spatial coordinates $c(x,y,z)$. The physical explanation of this would be that $c(x,y,z)$ determines the calcium carbonate concentration in a time when the rock surface that is continually in motion traverses the spatial point determined by the x,y,z coordinates. Naturally this time can be determined as written in (2). The sought for function shall be than

$$c = c(x, y, z), \qquad \text{while} \quad t = F(x, y, z). \qquad (34)$$

The benefit of (34) is that it contains only coordinates and no time (at least not explicitly) offering a possibility to determine the calcium carbonate concentration of the solution similarly to (1) independently of the position or type of the coordinate system, that is: in invariant form:

$$c = c(\mathbf{r}), \qquad \text{while} \quad t = F(\mathbf{r}). \qquad (35)$$

Differing from (31), (35) is not a two- but a three spatial dimensional function. Its explanation: $c(\mathbf{r})$ determines the concentration in the moment when the rock-surface traverses the point determined by the $\mathbf{r}$ vector. It will be revealed in the followings that this type of writing enables the creation of much more general equations than its creation observing concentration like in (33).

Relations between the concentration of calcium carbonate in the solution, the volume of rainfall and the density of the material flux of limestone shall be sought. The fact (according to observations) that the shifting of the limestone surface is slower by magnitudes than the velocity of the flow on it. (The rock dissolves 1-2 millimeters annually while the water make that distance in seconds.)

It shall be studied, how the concentration of the water varies at various points of the rock surface. To this purpose an infinitesimally small material volume of A base area and h height shall be pointed out. (*Fig. 4*)

The concentration of the solution (as concentration is defined):

$$c = \frac{m_K}{V}, \qquad (36)$$

where m_K is the mass of calcium carbonate in the pointed out V volume. When this volume is shifting along the surface, its concentration changes because on the one hand the mass of the dissolved calcium carbonate in it changes on the other hand it is diluted by the rainfall recharge. Because the material volume shifts along with the water particles, its shift in Δt time is

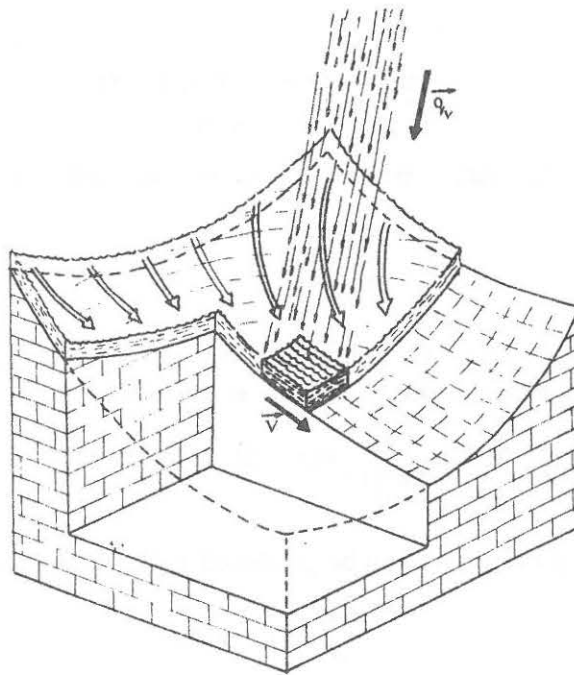$$\Delta \mathbf{r} = \mathbf{v} \cdot \Delta t. \qquad (37)$$

Figure 4. The position of the material volume cut from the water flowing on the rock surface

If the material volume was in the vicinity of the point determined by the **r** position vector in the t moment then it will obviously be at the point **r** + Δ**r** after the passing of Δ*t* time. According to (35) the concentration of the solution is different in the **r** + Δ**r** point as it was in point **r**, its value will change by

$$\Delta c = c(\mathbf{r} + \Delta \mathbf{r}) - c(\mathbf{r}) \tag{38}$$

Δ*c* compared with the elapsed time Δ*t* (knowing that according to (37) is Δ**r** = **v** · Δ*t*) the velocity of the change of the concentration is resulted:

$$\frac{\Delta c}{\Delta t} = \frac{c(\mathbf{r} + \mathbf{v} \cdot \Delta t) - c(\mathbf{r})}{\Delta t} \quad . \tag{39}$$

The value of **v** shall be the product of its absolute value and the **e**$_V$ direction of the flow:

$$\mathbf{v} = v \cdot \mathbf{e}_V , \tag{40}$$

thus

$$\frac{\Delta c}{\Delta t} = \frac{c(\mathbf{r} + v \cdot \Delta t \cdot \mathbf{e}_V) - c(\mathbf{r})}{\Delta t}. \tag{41}$$

The numerator and denominator of (41) multiplied by *v*, thus:

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$$\frac{\Delta c}{\Delta t} = v \cdot \frac{c(\mathbf{r} + v \cdot \Delta t \cdot \mathbf{e}_V) - c(\mathbf{r})}{v \cdot \Delta t}. \tag{42}$$

Obviously $v \cdot \Delta t = \Delta r$, where Δr is the absolute value of $\Delta \mathbf{r}$. As a consequence

$$\frac{\Delta c}{\Delta t} = v \cdot \frac{c(\mathbf{r} + \Delta r \cdot \mathbf{e}_V) - c(\mathbf{r})}{\Delta r}. \tag{43}$$

If $\Delta t \to 0$, than Δr converges to zero too, so the (43) formula is nothing else but the deviate of c in the $\mathbf{e}_V$ direction

$$\lim_{\Delta t \to 0} \frac{\Delta c}{\Delta t} = \lim_{\Delta r \to 0} \left\{ v \cdot \frac{c(\mathbf{r} + \Delta r \cdot \mathbf{e}_V) - c(\mathbf{r})}{\Delta r} \right\} = v \cdot \frac{\partial c}{\partial \mathbf{e}_V} \tag{44}$$

The directional derivative can be produced as the gradient of c

$$\frac{\partial c}{\partial \mathbf{e}_V} = \mathbf{e}_V \cdot \operatorname{grad} c, \tag{45}$$

or

$$\frac{\Delta c}{\Delta t} = \mathbf{v} \cdot \mathbf{e}_V \cdot \operatorname{grad} c. \tag{46}$$

On the other hand according to (40) $\mathbf{v} \cdot \mathbf{e}_V = \mathbf{v}$, so finally the function:

$$\frac{dc}{dt} = \mathbf{v} \cdot \operatorname{grad} c \tag{47}$$

has been achieved for the change of the concentration.

The derivative written in (47) indicates the so called material derivative, because it has been supposed that the material volume in question shifts together with the liquid.

The change of the dissolved calcium carbonate concentration in the liquid-film can be produced from (36) as well:

$$\frac{dc}{dt} = \frac{d}{dt} \left\{ \frac{m_K}{V} \right\}. \tag{48}$$

The derivation of the quotient on the right side executed:

$$\frac{d}{dt} \left\{ \frac{m_K}{V} \right\} = \frac{1}{V} \cdot \frac{dm_K}{dt} - \frac{m_K}{V^2} \cdot \frac{dV}{dt}, \tag{49}$$

After reduction and insertion to (48) comes to this form

$$\frac{dc}{dt} = \frac{1}{V} \cdot \frac{dm_K}{dt} - \frac{c}{V} \cdot \frac{dV}{dt} \qquad (50)$$

The individual factors in (50) shall be determined.
Obviously

$$V = A \cdot h. \qquad (51)$$

The time derivative of m_K is equal to the change per unit time of the volume of calcium carbonate in the material volume, that change can be originated only in the dissolution of the limestone surface.

As from the limestone surface a q_K quantity of rock material gets into the solution in a unit time, from an A surface A time that much in unit time, that is:

$$\frac{dm_K}{dt} = A \cdot q_K \qquad (52)$$

quantity will be dissolved.

The time derivative of V in accordance with the concept of the derivative is the change of the volume of the water in unit time, that is, equal with the rainfall on A area in unit time. If the density of the volume flux is indicated with $\mathbf{q}_V$ and the density of water with ρ_V, then the derivative of V will be

$$\frac{dV}{dt} = \frac{|\mathbf{q}_V|}{\rho_V} \cdot A \cdot \cos \alpha \qquad (53)$$

Where $\cos\alpha$ expresses that from the point of view of the new water supply only the projection of A surface to the direction of the rainfall shall be considered, because the rain does not fall on the surface at right angle but at an α angle.

Indicating the direction of the rainfall with $\mathbf{n}_V$, $\cos\alpha$ can be written as a scalar product:

$$\cos \alpha = \mathbf{n} \cdot \mathbf{n}_V, \qquad (54)$$

where $\mathbf{n}$ is the normal of the rock surface. Inserting (54) to (53):

$$\frac{dV}{dt} = \frac{|\mathbf{q}_V|}{\rho_V} \cdot A \cdot \mathbf{n} \cdot \mathbf{n}_V. \qquad (55)$$

Naturally $|\mathbf{q}_V|\mathbf{n}_V = \mathbf{q}_V$, thus

$$\frac{dV}{dt} = \frac{A}{\rho_V} \cdot \mathbf{q}_V \cdot \mathbf{n}. \tag{56}$$

Inserting the (52) and (56) equations to (50):

$$\frac{dc}{dt} = \frac{1}{V} \cdot A \cdot q_K - \frac{c}{V} \cdot \frac{A}{\rho_V} \cdot \mathbf{q}_V \cdot \mathbf{n}. \tag{57}$$

Writing V from (51) to (57) and executing the possible reductions:

$$\frac{dc}{dt} = \frac{q_K}{h} - \frac{c}{\rho_V \cdot h} \cdot \mathbf{q}_V \cdot \mathbf{n}. \tag{58}$$

Finally inserting (58) to the left side of (47) the below differential equation is resulted determining the calcium carbonate concentration dissolved in the water:

$$\mathbf{v} \cdot \operatorname{grad} c = \frac{q_K}{h} - \frac{c}{\rho_V \cdot h} \cdot \mathbf{q}_V \cdot \mathbf{n}. \tag{59}$$

From (59) we can eliminate $\mathbf{n}$ with respect of (20):

$$\mathbf{v} \cdot \operatorname{grad} c = \frac{q_K}{h} - \frac{c}{\rho_V \cdot h} \cdot \mathbf{q}_V \cdot \frac{\operatorname{grad} F}{|\operatorname{grad} F|}. \tag{60}$$

The Chemical Equation of Dissolution

According to the most simple model of solution the dissolution is the quicker if the difference between the saturation c_e and de facto c concentration is the bigger (*DREYBROT, W.* 1988). As a consequence the more calcium carbonate gets into the material volume showed in *Figure 4* the aggressivity of the solution is the bigger (that is: c_e–c difference is the bigger), or the dissolution happens on a bigger A surface. In a formula:

$$\frac{dm_K}{dt} = k \cdot (c_e - c) \cdot A, \tag{61}$$

where k is a constant characteristic to the velocity of dissolution. Calculating with equation (36) the

$$A \cdot q_K = k \cdot (c_e - c) \cdot A \tag{62}$$

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equation is resulted. Using this one the

$$q_K = k \cdot (c_e - c) \qquad (63)$$

equation is gained for the relation of the concentration of the solution and the velocity of dissolution.

This equation along with the consideration of (32) creates a direct relation between the concentration and the shape of the surface:

$$\frac{\rho_K}{|\text{grad } F|} = k \cdot (c_e - c). \qquad (64)$$

This paper does not describe the ways of the measurement of the c_e and k values, this has been worked out in other speleological studies in relatively good details. It has to be emphasized anyway that the k constant helps to involve the role of the mixing corrosion, the atmospheric temperature and carbon dioxide content, the increase of the partial pressure of carbon dioxide under extended snow covering.

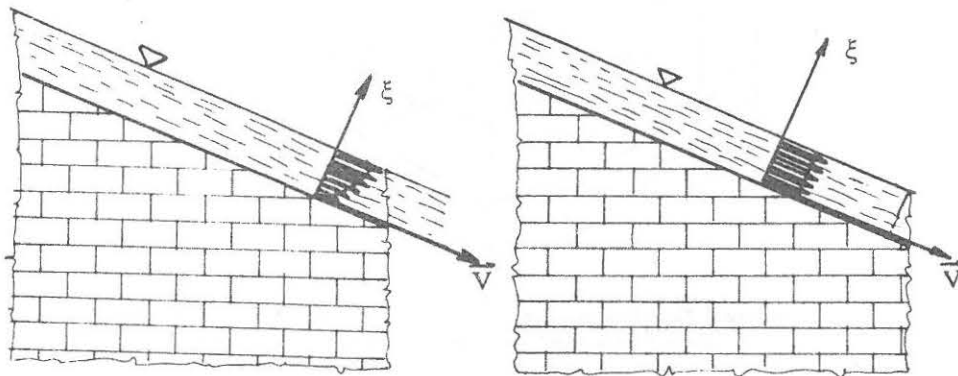**The Flow Velocity of Water in the Fluid-Film**

Figure 5. The actual and the equalized velocity profile of the liquid film

As the thin layer of water flowing on the rock surface behaves as a fluid with friction, its flow velocity gradually increases with the distance from the rock surface. The velocity of the flow is however meaningless concerning the material-transport. The real flow can be substituted with a fictive flow with uniform velocity-profile (*Fig. 5*) and with a discharge identical with the real flow.

It is known from the theory of frictional fluids that the discharge of a B wide h deep, B slope canal is

$$Q = \frac{\rho_V \cdot g \cdot h^3 \cdot B \cdot \sin\beta}{3 \cdot \eta},$$
(65)

where ρ_V is the density of water, η is the coefficient of dynamic viscosity (*SZUNYOGH, G.* 1995). The linear velocity profile shall produce the same discharge, thus:

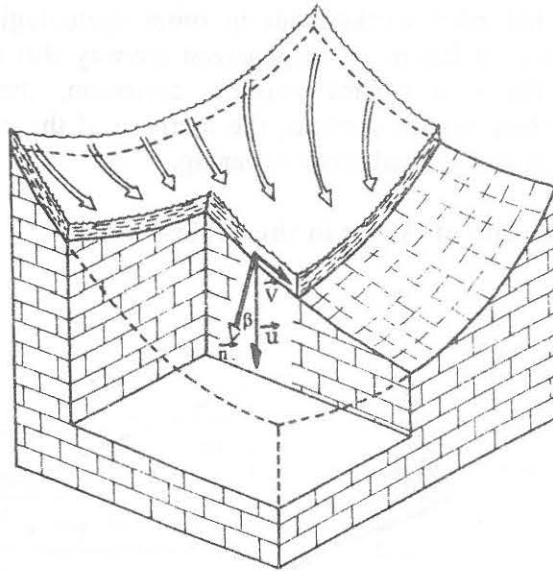$$Q = v \cdot h \cdot B.$$
(66)

Figure 6. The relative position of the flow velocity of the liquid, the normal of the rock surface and the vectors of gravitational acceleration

Equalizing the (65) and (66) equations for the average flow velocity the following formula is resulted:

$$v = \frac{\rho_V \cdot g \cdot h^2 \cdot \sin\beta}{3 \cdot \eta}$$
(67)

The sin$B$ coefficient occurring in (67) hints that v can be taken as the surface-dip oriented component of some vertical **u** vector (see *Figures 6* and *7*).

This particular **u** can be created by the help of **g** gravitational acceleration (**g** obviously vertical):

$$\mathbf{u} = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \mathbf{g}. \tag{68}$$

According to the conditions stated at the beginning of this paper the direction of the flow is determined by the slope direction of the surface. The gradient lines are drawn up by the plane defined by the normal of the rock surface and the gravitational acceleration and curve of intersection of this plane with the rock surface. Figure 7 shows the section of the surface of the limestone body along the above defined plane.
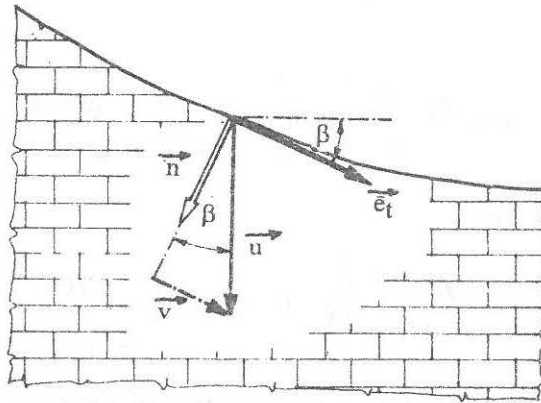

Figure 7. The section of the limestone formation along the plane determined by the normal of the rock surface and the vector of the gravitational acceleration

It can be seen on *Figure 7* that **v** is linear a combination of the **u** and **n** vectors

$$\mathbf{v} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n}) \cdot \mathbf{n}, \tag{69}$$

that has an absolute value (after *Fig. 7*):

$$|\mathbf{v}| = |\mathbf{u}| \cdot \sin\beta = \left| \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \mathbf{g} \right| \cdot \sin\beta = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot g \cdot \sin\beta \tag{70}$$

It shows that the right side of (70) is really identical with the right side of (67), thus the guess has been proved that v can really be expressed with the help of a vertical **u** vector.

Inserting the expression of **u** in (68) to (69) it is resulted that

$$\mathbf{v} = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \mathbf{g} - \left(\frac{\rho_V \cdot h^2}{3 \cdot \eta}\mathbf{g} \cdot \mathbf{n}\right) \cdot \mathbf{n}. \tag{71}$$

Reducing for the scalars:

$$\mathbf{v} = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \left[\mathbf{g} - (\mathbf{g} \cdot \mathbf{n}) \cdot \mathbf{n}\right]. \tag{72}$$

It shall be taken into account that

$$\mathbf{n} = \frac{\operatorname{grad} F}{|\operatorname{grad} F|}, \tag{73}$$

thus

$$\mathbf{v} = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \left[\mathbf{g} - \left(\mathbf{g} \cdot \frac{\operatorname{grad} F}{|\operatorname{grad} F|}\right) \cdot \frac{\operatorname{grad} F}{|\operatorname{grad} F|}\right]. \tag{74}$$

By further reducing:

$$\mathbf{v} = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \left[\mathbf{g} - (\mathbf{g} \cdot \operatorname{grad} F) \cdot \frac{\operatorname{grad} F}{(\operatorname{grad} F)^2}\right]. \tag{75}$$

It can be seen that the last component of the rights side (according to (23) is identical shift velocity of the surface:

$$\mathbf{w} = \frac{\operatorname{grad} F}{(\operatorname{grad} F)^2}, \tag{76}$$

So finally the equation developed for the flow velocity of the fluid-film is:

$$\mathbf{v} = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \left[\mathbf{g} - (\mathbf{g} \cdot \operatorname{grad} F) \cdot \mathbf{w}\right]. \tag{77}$$

The Thickness of the Fluid-Film

The thickness of the fluid film flowing on the rock surface can be determined using the law of conservation of mass. For this purpose a section of conduit, relatively narrow compared to its length will be pointed out.

According to the law of conservation of mass the mass of water flowing into and out of the conduit within unit time will be identical. Let (see *Fig. 8*) the width of the conduit b, its length s and its height (that is obviously the same as the height of the fluid film) h_1 at the point of entry and h_2 at the point of exit. The flow velocity shall be v_1 and v_2. The $\mathbf{r}_1$ position vector shall point at the entry point, $\mathbf{r}_2$ vector to the exit point. If the conduit is short enough the curvature of the flow line is negligible and the conduit can be approached by a straight line, thus

$$\mathbf{r}_2 \cong \mathbf{r}_1 + \mathbf{s}, \qquad (78)$$

where $\mathbf{s}$ is a minute vector that is parallel with the flow lines and that has an absolute value that is the same as the length of the conduit. The water balance of this conduit shall be examined, that is that the entering and exiting of volumes of water shall be studied.
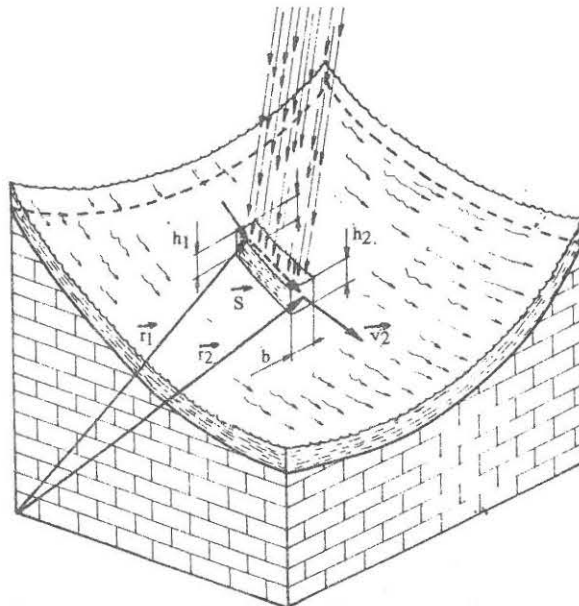
Figure 8. The position of the conduit necessary for the application of the law of mass conservation

There is gain originated from the recharge at one end of the conduit and on the top of it from the rainfall. Thus the mass of recharge water is:

$$Q_{BE} = \rho_V \cdot v_1 \cdot h_1 \cdot b + \mathbf{q}_V \cdot \mathbf{n} \cdot s \cdot \quad . \qquad (79)$$

Discharge at the other end is responsible for the loss at the other end of the conduit. The mass of the discharge:

$$Q_{KI} = \rho_V \cdot v_2 \cdot h_2 \cdot b . \tag{80}$$

According to the law of conservation of mass

$$Q_{BE} = Q_{KI}, \tag{81}$$

that after equalizing (79) and (80) results

$$\frac{v_2 \cdot h_2 - v_1 \cdot h_1}{s} \cong \frac{\mathbf{q}_V \cdot \mathbf{n}}{\rho_V} \tag{82}$$

Approximation is the more accurate s is the more short, because the flow lines can the better substituted by straight lines.

v_1, v_2, h_1 and h_2 can be produced as the values of the $v(\mathbf{r})$ and $h(\mathbf{r})$ functions at $\mathbf{r}_1$ and $\mathbf{r}_2$

$$v_1 = v(\mathbf{r}_1), \tag{83}$$

$$v_2 = v(\mathbf{r}_2), \tag{84}$$

$$h_1 = h(\mathbf{r}_1), \tag{85}$$

$$h_2 = h(\mathbf{r}_2). \tag{86}$$

Inserting these expressions to (82) and observing that

$$\mathbf{r}_2 \cong \mathbf{r}_1 + \mathbf{r} \tag{87}$$

it follows that

$$\frac{v_2 \cdot h_2 - v_1 \cdot h_1}{s} \cong \frac{v(\mathbf{r}_1 + \mathbf{s}) \cdot h(\mathbf{r}_1 + \mathbf{l}) - v(\mathbf{r}_1) \cdot h(\mathbf{r}_1)}{s} . \tag{88}$$

It can be noted that on the right sight of (88) (in case $s \to 0$) the s-direction derivative of product $v(\mathbf{r}) \cdot h(\mathbf{r})$ appears, thus

$$\lim_{s \to 0} \frac{v_2 \cdot h_2 - v_1 \cdot h_1}{s} = \frac{\partial}{\partial s} \{ v(\mathbf{r}) \cdot h(\mathbf{r}) \} . \tag{89}$$

The directional derivative in question can be produced as the gradient of the $v(\mathbf{r}) \cdot h(\mathbf{r})$ product:

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$$\frac{\partial}{\partial s}\{v(\mathbf{r}) \cdot h(\mathbf{r})\} = \mathbf{e}_S \cdot \mathrm{grad}(v \cdot h), \tag{90}$$

where $\mathbf{e}_s$ stands for the $\mathbf{s}$ direction unit vector. According to the above described the direction of $\mathbf{e}_s$ is identical $\mathbf{v}$ direction of the flow.

The $\mathrm{grad}(v \cdot h)$ expression shall be developed by the rules of gradient making (*FRANK, Ph.— MIESES, R.* 1967):

$$\mathbf{e}_S \cdot \mathrm{grad}\,(v \cdot h) = \mathbf{e}_S \cdot v \cdot \mathrm{grad}\,h + \mathbf{e}_S \cdot h \cdot \mathrm{grad}\,v. \tag{91}$$

The first coefficient of the first term can be pooled like this:

$$\mathbf{e}_S \cdot v = \mathbf{v}, \tag{92}$$

as $\mathbf{e}_s$ is the unit vector pointing in the direction of $\mathbf{v}$. The second term can be shaped like this:

$$\mathbf{e}_S \cdot h \cdot \mathrm{grad}\,v = \mathbf{e}_S \cdot h \cdot \mathrm{grad}\,\sqrt{\mathbf{v}^2} = \frac{\mathbf{e}_S \cdot h}{2\sqrt{\mathbf{v}^2}} \cdot \mathrm{grad}\,(\mathbf{v} \cdot \mathbf{v}) =$$

$$= \frac{\mathbf{e}_S \cdot h}{2 \cdot \mathbf{v}} \cdot (\mathbf{v} \cdot \mathrm{div}\,\mathbf{v} + \mathbf{v} \cdot \mathrm{div}\,\mathbf{v}) = \frac{\mathbf{e}_S \cdot h}{\mathbf{v}} \cdot \mathbf{v} \cdot \mathrm{div}\,v. \tag{93}$$

As the water is an incompressible fluid, the divergence of its velocity field is zero, thus

$$\mathrm{div}\,\mathbf{v} = 0, \tag{94}$$

consequently

$$\mathbf{e}_S \cdot \mathrm{grad}(v \cdot h) = \mathbf{e}_S \cdot v \cdot \mathrm{grad}\,h. \tag{95}$$

Inserting (95) to (90) and that to (89) the following differential equation is produced for the thickness of the water-film.

$$\mathbf{v} \cdot \mathrm{grad}\,h = \frac{\mathbf{q}_V \cdot \mathbf{n}}{\rho_V} \tag{96}$$

It can be red in (96) that if rain falls on the rock, that is $\mathbf{q}_V \neq 0$, then the thickness of the fluid-film increases in the direction of the slope lines because $\mathbf{v} \cdot \mathrm{grad}\,h \geq 0$. The change of the fluid-film thickness can occur in the case of $\mathbf{q}_V = 0$ too because the flow lines of the flowing water can contract or spread out.

It can be deducted from (96) that if the direction of the rainfall and the normal of the rock surface are at a small angle than the $\mathbf{q}_V \cdot \mathbf{n}$ scalar product is small and the pace of thickening of the fluid film is smaller.

Summary

The equation system of the dissolution of limestone surface uncovered by soil (free) is composed of the following equations:

1. Relation between the normal of the rock surface and the function describing the shape of the surface:

$$\mathbf{n} = \frac{\operatorname{grad} F}{|\operatorname{grad} F|};$$
(97)

2. Relation between the sinking velocity of the rock surface and the function describing the shape of the surface:

$$\mathbf{w} = \frac{\operatorname{grad} F}{(\operatorname{grad} F)^2};$$
(98)

3. Relation between the density of the material flux of the dissolved limestone and the sinking velocity of the rock surface:

$$q_K = \mathbf{w} \cdot \mathbf{n} \cdot \rho_K;$$
(99)

4. Relation between the calcium carbonate concentration of the solution flowing on the surface, the density of the material flux of the dissolve limestone, the thickness of the fluid-film and rainfall:

$$\mathbf{v} \cdot \operatorname{grad} c = \frac{q_K}{h} - \frac{c}{\rho_V \cdot h} \cdot \mathbf{q}_V \cdot \mathbf{n};$$
(100)

5. Relation between the calcium carbonate concentration of the solution flowing on the surface and the density of the material flux of the dissolved limestone:

$$q_K = k \cdot (c_e - c);$$
(101)

6. Relation between the flow velocity, the thickness of the fluid-film and the spatial position of the rock surface (its normal):

$$\mathbf{v} = \frac{\rho_V \cdot h^2}{3 \cdot \eta} \cdot \left[\mathbf{g} - (\mathbf{g} \cdot \mathbf{n}) \cdot \mathbf{n} \right];$$
(102)

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7. Relation between the thickness of the fluid-film, the flow velocity, the rainfall and the spatial position of the rock surface:

$$\mathrm{div}(h\mathbf{v}) = -\frac{\mathbf{q}_V \cdot \mathbf{n}}{\rho_V}.\qquad(103)$$

Unknown quantities in the equations:
1. The $t=F(x,y,z)$ function describing the rock surface;
2. The normal of the rock surface ($\mathbf{n}$);
3. The sinking velocity of the rock surface ($\mathbf{w}$);
4. The mass of rock dissolved from a unit surface of limestone in unit time (that is: the density of the material flux of the dissolved limestone) (q_K);
5. The calcium carbonate concentration of the solution flowing on the surface (c);
6. The flow velocity of water in the fluid-film ($\mathbf{v}$);
7. The thickness of the fluid-film (h)

The number of unknown quantities and equations. The (97)-(103) equation system consists of 3 vectorial and 4 scalar equations. 3 vectorial and 4 scalar values are sought for thus the number of equations and unknown quantities are equal.

Boundary conditions. The thickness of the fluid-film shall be known at the boundary of the studied area where the water enters it and also the concentration of the entering water shall be known. The rest of the boundary conditions can be derived in this course.

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