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THE THEORETICAL-PHYSICAL STUDY OF THE PROCESS OF KARREN DEVELOPMENT

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Abstract: the results of a theoretical-physical study of the process of karren development is presented in this paper. Referring to former studies the equation system of the karstification of a sloping limestone terrain without pit formation is written considering the hydrodynamic, chemical and morphological rules of the karstification processes of a limestone rock surface. The differential geometric correlations that are necessary for the mathematical description of rock surfaces that change their shape in time are determined and the quantitative relations of physicochemical processes influencing the changes of their parameters in time are described. The basically sought for function will be the z(x,y,t) one that determines the shape of the karstified ground surface, but that demands the computation of the flow rate of the water flowing on the limestone surface as well as the calcium carbonate concentration in the water and the thickness of the liquid film. The computer solution of the algorithm of the derived partial differential equation is also presented.

Preliminaries

The Karst Research Group of the "Berzsenyi Dániel" College, Department of Geography published the general equation system of the karstification of an exposed limestone ground surface not covered by soil (*SZUNYOGH, G.* 1994). The "ultimate target" of this equation system was

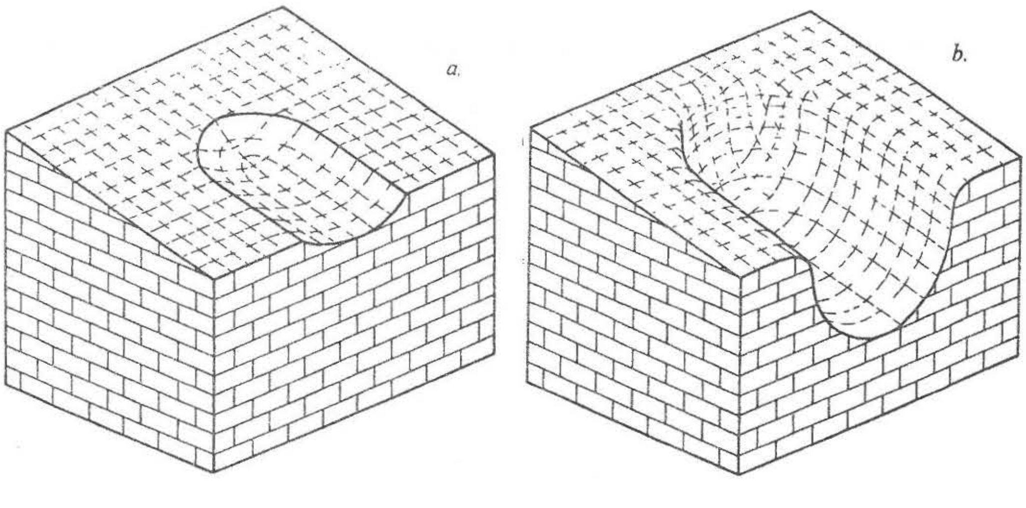

Fig. 1, a: the initial shape at the t_0 moment of the studied limestone surface uncovered by soil, b: the shape of the same limestone surface changed by karst corrosion in a later t moment (in the knowledge of the relevant physical, chemical and geological principles as well as the necessary preliminary and boundary conditions) to develop theoretically the mathematical determination of the future shape of a limestone surface that has been known at the beginning thus to support the principles of classic karst morphology (*JAKUCS, L.* 1971) in a physicochemical way (*Fig. 1*). This mathematical modeling apparently does not substitute but only adds to classic karst morphological studies enabling the checking of hypotheses (based on physicochemical principles) that are very slow processes that can not be studied by tests like the forecast of the looks of a rocky surface after a number of centuries or millennia; the study of karstification processes at conditions (hydrological, climatological etc.) that aren't there to study at the present; the forecast and quantitative study of global changes of the environment and such.

To achieve these a partial differential equation in several variables was written of which the solution is an $F(\mathbf{r})$ function that determines at what t time will the rock surface ever changed by solution pass the point in space that is characterized by the $\mathbf{r}$ position vector. The limestone surface is sought for in the form of:

$$t = F(\mathbf{r}) \tag{1}$$

It can be derived (*SZUNYOGH, G.* 1995a) that the $\mathbf{w}$ velocity vector of the displacement (denudation) [m/s] is

$$\mathbf{w} = \frac{\mathbf{n}}{|\mathrm{grad}\, F|}, \tag{2}$$

where $\mathbf{n}$ is the unit vector perpendicular at the rock surface (the normal of the limestone surface)

It can be computed from the measure of denudation what volume of limestone was removed from a unit surface area in unit time, thus the so called q_k mass-flux density of the limestone [kg/m^{2}s] can be expressed:

$$q_k = -\rho_k \mathbf{w} \cdot \mathbf{n}, \tag{3}$$

where the ρ_k is the density of the limestone [kg/m^3].

Starting from the mass conservation principle it can be proved that the

$$\mathbf{v} \cdot \mathrm{grad}\, c = \frac{q_k}{h} + \frac{c}{\rho_V h} \mathbf{q}_v \mathbf{n} \tag{4}$$

relation exists between mass-flux density of the removed limestone and the concentration of calcium carbonate in the water, where $\mathbf{v}$ is the velocity of the water flow on the limestone surface [m/s], h is the thickness of the liquid layer [m] and q_v the rainfall that replenishes the solvent, or the volume of rainfall on a unit area in unit time [kg/m^2s].

The velocity of the water flow v is determined by the gravity and the friction force (through the **g** gravity and η viscosity factor), that derived from the Navier-Stokes formula (*FRANK, Ph.—MIESES, R.* 1967):

$$\mathbf{v} = \frac{\rho_V h^2}{3\eta}\left[\mathbf{g} - (\mathbf{g} \cdot \mathbf{n})\mathbf{n}\right]. \qquad (5)$$

The mass conservation principle is valid for the water alone too that derived from the equation of continuity takes the form:

$$\oiint_{(A)} (\rho_v \mathbf{v} - \mathbf{q}_v) d\mathbf{A} = 0 \qquad (6)$$

where A is a closed surface of a specified static volume in the water film, p_v the density of the water (kg/m^3).

The last equation of the dissolution reflects the chemical principles of karst development expressing that the more limestone is turned into solution as the water is more aggressive, that is, the more is the difference between the maximal measure of solubility and the calcium carbonate that is actually in the solution:

$$q_k = k(c_e - c). \qquad (7)$$

where k is the constant of the reaction velocity of the dissolution [m/s] (*DREYBROT, W.* 1988).

The (2)-(7) equation system is general in the sense that its validity is independent of the choice of a coordinate system thus it can be flexibly fitted to the system of coordinates that is most suitable to the geometry of the modeled karstic phenomenon. This generality comes together with some disadvantage: the published equations (in their original form) are not suitable for the solution of any specific problem, but first they have to be adjusted to some coordinate system fit to the specific task.

In the present paper this adjustment will be done to the Cartesian coordinate system, because that's the one most fit to the mathematical analysis of the karstic processes occurring on alpine limestone surfaces.

The Equation System of the Dissolution of Open Limestone Surface with the Application of Cartesian coordinate system

For the specification in space of the limestone surface such an orthogonal coordinate system shall be applied that has horizontal x and y axes and an upwards pointing z axis. (*Fig. 2*).

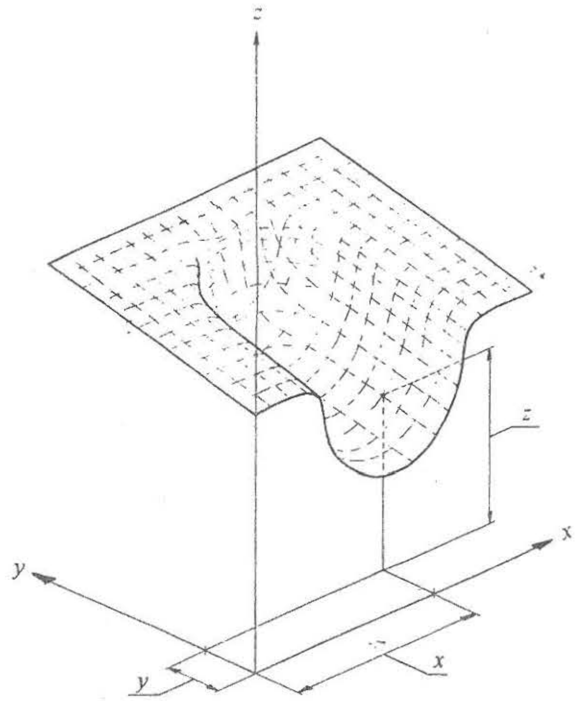

Fig. 2: The position of the coordinate system that describes the location of the limestone surface at any moment

All the unknowns in the (2)-(7) equation are the functions of x and y space coordinates and t time. The determination of these functions is aimed at, the seeking of a

$$z = f(x,y,t) \tag{8}$$

function that is the mathematical specification of the limestone surface.

The Normal Vector of the Limestone Surface in the Cartesian Coordinate System

As the normal of the limestone surface occurs more than once in the described equation aystem first **n** shall be determined as the derivative of the *z(x,y,t)* function that determines the surface (*Fig. 3*). The gradient in the x direction will be α, at y it will be β. For the components of the normal vector: n_x, n_y and n_z (*Fig. 4*) it can be written:

$$n_x = -n_z \operatorname{tg} \alpha, \tag{9}$$

$$n_y = -n_z \operatorname{tg} \beta, \tag{10}$$

and

$$n_x^2 + n_y^2 + n_z^2 = 1. \tag{11}$$

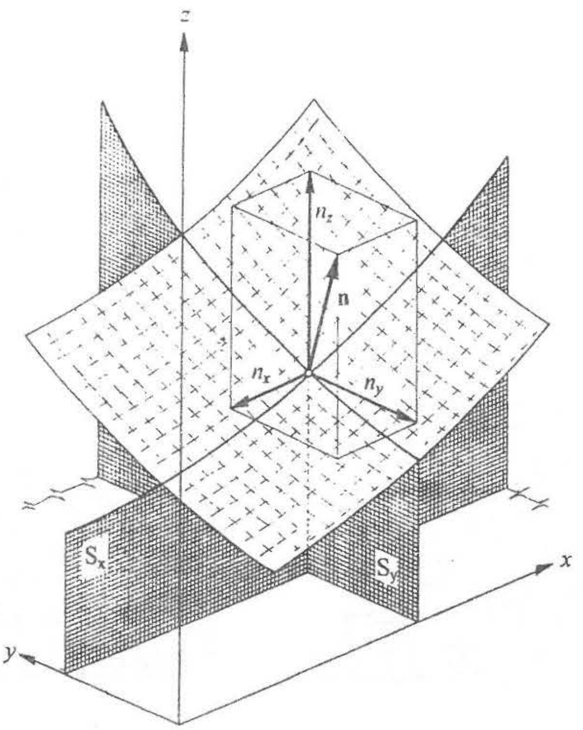

Fig. 3: The position of the so called normal vector that is at right angle at the rock surface and its resolution to components

The tgα and tgβ quantities will be equal with the gradient of the limestone surface in the x and y directions so they can be expressed by the partial derivatives of the function determining the rock surface:

$$\mathrm{tg}\,\alpha = \frac{\partial z}{\partial x}, \tag{12}$$

$$\mathrm{tg}\beta = \frac{\partial z}{\partial y}. \tag{13}$$

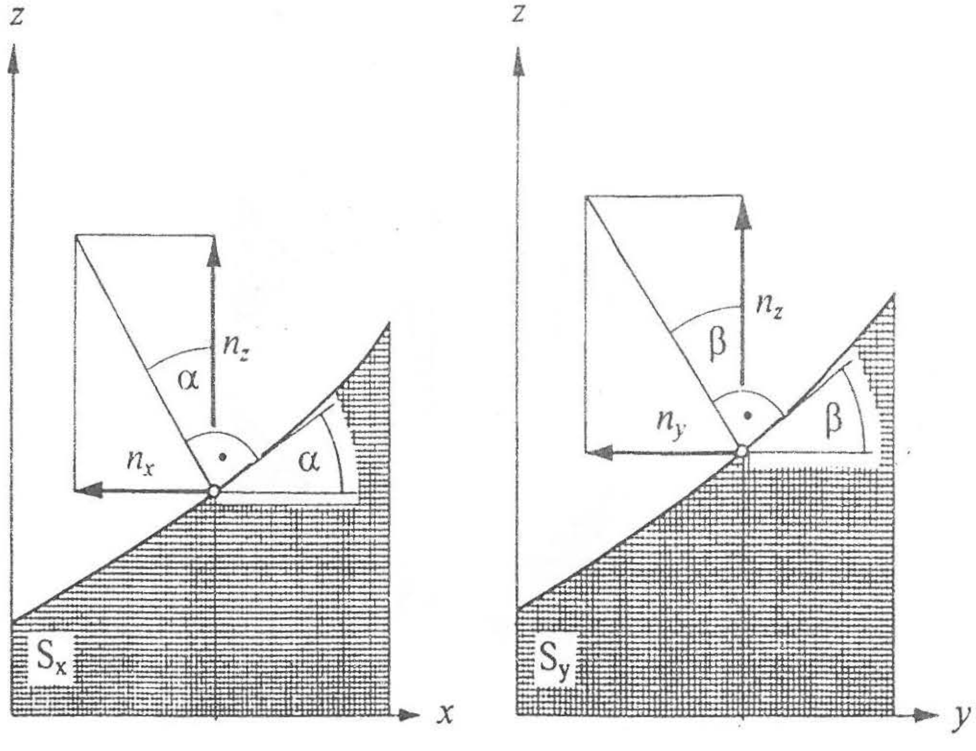

Fig. 4: The n_x, n_y and n_z components of the normal vector of the rock surface drawn in the S_x and S_y plane sections as showed in Fig. 3

The solutions of the (9)—(13) equations regarding n_x, n_y and n_z are:

$$n_x = \frac{-\dfrac{\partial z}{\partial x}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}, \tag{14}$$

$$n_y = \frac{-\dfrac{\partial z}{\partial y}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}, \tag{15}$$

and

$$n_z = \frac{1}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}. \tag{16}$$

The (14)—(16) equations enable the calculation of the components in the Cartesian coordinate system of the unit vector perpendicular at the rock surface in the knowledge of the equation of the surface.

The Velocity of the Denudation of the Surface

As a result of karst corrosion the rock surface is shifting, sinking very slowly but continuously with a **w** velocity. The **w** is understood as the thickness of the surface layer that is removed by solution in unit time. Its direction is at right angle to the rock surface and it points to the interior of the fresh limestone (*Fig. 5*).

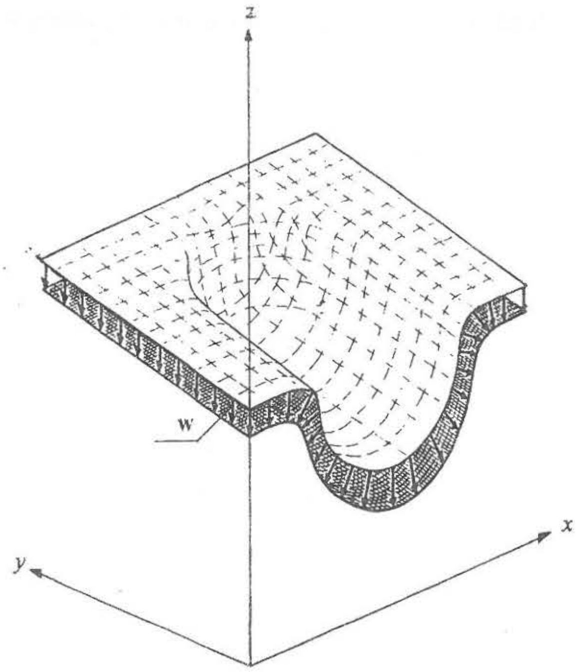

Fig. 5: The specification of the velocity vector of denudation

Writing the gradient expression in the formula of w to a coordinate form the

$$\frac{1}{|\text{grad}\, F|} = \frac{1}{\sqrt{\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2}} \tag{17}$$

function is received that considering the rules of derivation of inverse functions can be transformed:

$$\frac{1}{|\text{grad}\, F|} = \frac{\frac{\partial z}{\partial t}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} \tag{18}$$

According to (2) for the determination of w (18) shall be multiplied with the normal of the surface that considering (14)—(16) produces the equations:

$$w_x = \frac{-\dfrac{\partial z}{\partial t}\dfrac{\partial z}{\partial x}}{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}, \tag{19}$$

$$w_y = \frac{-\dfrac{\partial z}{\partial t}\dfrac{\partial z}{\partial y}}{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}, \tag{20}$$

and

$$w_z = \frac{\dfrac{\partial z}{\partial t}}{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1} \tag{21}$$

The absolute value of the velocity will be

$$|\mathbf{w}| = \sqrt{w_x^2 + w_y^2 + w_z^2}\,. \tag{22}$$

that after the execution of the assigned operations take the form:

$$|\mathbf{w}| = \frac{\left|\dfrac{\partial z}{\partial t}\right|}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}\,. \tag{23}$$

The Mass-Flux Density of the Dissolving Rock

The expression (3) for the mass-flux density considering the (14)—(15) and (20)—(22) expressions and after the scalar composition and ordering equation (3) gets into the form:

$$q_k = -\rho_K \frac{\dfrac{\partial z}{\partial t}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}} \tag{24}$$

It is apparent from equation (24) that when the rock is denuding $q_k \geq 0$ than $\frac{\partial z}{\partial t} \leq 0$, thus the surface gets to ever lower elevation, it is sinking.

The Chemical Equation of the Dissolution of Calcium Carbonate

The chemical equation of the dissolution expresses how much more limestone transfers to solution in unit time (how much more is the mass-flux density of the dissolving calcium carbonate) when the water is the more aggressive, that is the difference between the actual calcium carbonate content of the water and the total limestone solubility, is the bigger. This relation is mathematically incorporated in (7) (*VERESS, M.—PÉNTEK, K.* 1990, 1992). Writing the mass-flux density defined by (24) to the left side a relation is resulted between the derivatives of the function defining the rock surface and the concentration of calcium carbonate in the solution.

$$-\rho_k \frac{\frac{\partial z}{\partial t}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} = k(c_e - c). \qquad (25)$$

Expressing $\frac{\partial z}{\partial t}$ from this an equation is resulted for the velocity of sinking of the rock surface:

$$\frac{\partial z}{\partial t} = -k \frac{c_e - c}{\rho_k} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}. \qquad (26)$$

It can be seen in (26) that the more aggressive is the water, (the difference between the equilibrium (c_e) and actual (c) calcium carbonate concentration is the bigger) the quicker is the denudation of the rock. It can be also seen that the gradient of the slope of the area (that is expressed by the $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ after (12) and (13) plays a boosting role in the velocity of the sinking of the surface.

The algebraic sign of the $\frac{\partial z}{\partial t}$ shall be examined considering (26) than can be done by the analysis of its constituents. The k, the first factor at the right side of the equation is a chemical constant, that is a positive number. The density of the rock (ρ_k) in the denominator is a positive number too.

The second factor in the numerator of (26) can not be negative, because that would represent an oversaturated solution that is impossible in the case of corrosion, that is:

$$c_e - c \geq 0. \tag{27}$$

The last factor in (26) is the positive root computed from the sum of squares:

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \geq 0. \tag{28}$$

It is clear from the above described that all values on the right side of (26) are positive thus their products are positive as well. But because (26) is completed with a (-) sign, it can be stated that

$$\frac{\partial z}{\partial t} \leq 0, \tag{29}$$

that is, the rock surface can not get any higher as a result of karst corrosion, it can only sink. (With the passing of time the elevation of the limestone surface becomes ever lower.) This is virtually apparent but the mathematical "reflection" of the well known principle proves the validity of the deductions.

The Spatial Development of the Calcium Carbonate Content of the Water Flowing on the Rock Surface

While the water flows on the surface it continually dissolves calcium carbonate. The velocity of dissolution is different at the various points of the rock surface as it is depending on numerous factors and first of all the quantity of calcium carbonate that has been previously dissolved in the water (c) and the velocity of the flow of the water film (*Fig.6*). This written in the Cartesian coordinate system:

$$\frac{\partial c}{\partial x} v_x + \frac{\partial c}{\partial y} v_y = \frac{q_k}{h} + \frac{c}{\rho_V h} \mathbf{q}_v \mathbf{n}, \tag{30}$$

where v_x and v_y are the x- and y components of the velocity vector of the water, h is the thickness of the water film on the rock surface, $\mathbf{q}_v$ is the quantity of rainfall. Thus (30) creates a relation between the flow velocity of the water, its chemical constitution, the volume of the corroded rock and the rainfall on the area.

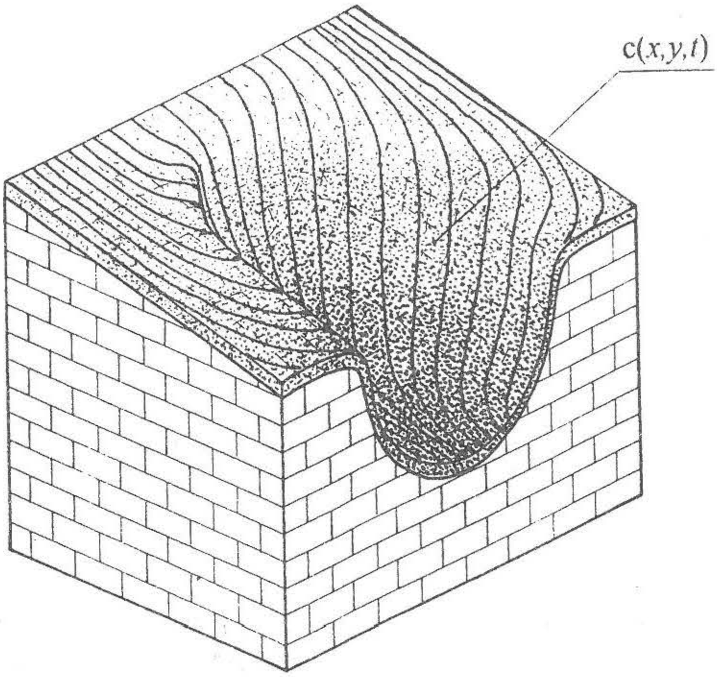

Fig. 6: The spatial distribution of calcium carbonate dissolved in the water film flowing on the rock surface

The components of the $\mathbf{q}_v$ vector in the Cartesian coordinate system

$$\mathbf{q}_v = \begin{cases} 0, \\ 0, \\ -q_v. \end{cases} \qquad (31)$$

where q_v is a positive number. Its unit is: kg/m^2s.

Substituting the form of q_k determined in (7) to the first factor in the right of (30) and $\mathbf{q}_v$ as it is determined in (31) to the second factor. Considering the (14)—(16) equations of the normal of the rock surface the assigned scalar composition shall be executed. At length the

$$\frac{\partial c}{\partial x} v_x + \frac{\partial c}{\partial y} v_y = k \frac{c_e - c}{h} - \frac{c}{\rho_V h} q_v \qquad (32)$$

equation is resulted for the spatial distribution of dissolved calcium carbonate.

(32) tells that the aggressivity (c_e-c) of the water flowing on the rock surface has a boosting effect on the concentration of calcium carbonate, because the water dissolves the limestone as long as the rainfall reduces the concentration (dilutes the solution) as it is written with a negative sign in the right of (32).

The Velocity of the Water Flowing on the Rock Surface

The flow of the water is caused by gravity, it is slowed by friction. The forces of inertia should be calculated with but in the relatively slowly flowing water film the letter can be neglected beside the former two. (*Fig. 7*). Naturally the forces of inertia shall be calculated with at high velocity flow in its full form on the left of the Navier-Stokes equation (*SZUNYOGH, G.* 1995b).

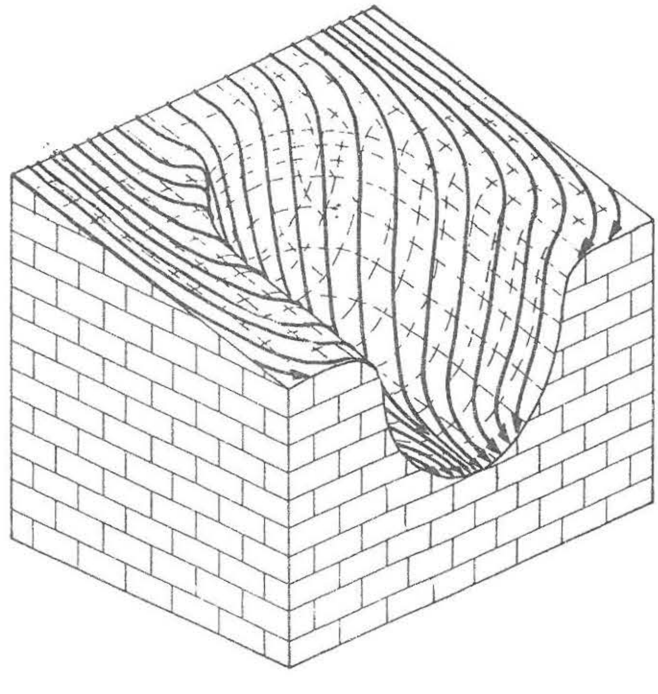

Fig. 7: The flow lines of the water film that covers the rock surface

In the (5) formula written for the velocity the **g** vector of the gravity occurs. As the gravity is apparently vertical and points downwards, it has only a *z*-directed vector component, that is:

$$\mathbf{g} = \begin{cases} 0, \\ 0, \\ -g, \end{cases} \tag{33}$$

where the absolute value of gravity is (10 m/s^2). Executing the **gn** scalar composition on the right side of (5) after the necessary ordering the following equations can be achieved for the vector components of the flow velocity of the water.

$$v_x = -\frac{\rho_V g h^2}{3\eta} \frac{\frac{\partial z}{\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \tag{34}$$

$$v_y = -\frac{\rho_V g h^2}{3\eta} \frac{\frac{\partial z}{\partial y}}{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}, \tag{35}$$

$$v_z = -\frac{\rho_V g h^2}{3\eta} \frac{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \tag{36}$$

The negative sign in the (34)—(36) formulas expresses that if the surface seen in the direction of the x or y rises, the water flows backwards, toward the origin, so:

$$\text{if } \frac{\partial z}{\partial x} \geq 0, \text{ then } v_x \leq 0,$$

or

$$\text{if } \frac{\partial z}{\partial x} \geq 0, \text{ then } v_y \leq 0. \tag{37}$$

The equations tell that if the surface is steep, so $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$ is large, the water flows with greater velocity. It is visible too that the flow velocity of the water increases in a quadratic way with the flow thickness. It can be deducted that the assumption that the flow in a thin water film is slow has been proved.

The Equation of the Thickness of the Liquid Film

For the determination of the thickness of the liquid film flowing on the limestone surface the continuity equation that expresses the mass conservation of the water can be applied in a way that for the closed A surface used for the application of equation (6) a minute tilted prism shall be taken that includes the full m thickness of the flow and its base is Δx x Δy. The integration in (6) performed and decreasing the values of Δx and Δy beyond all limits (converging them to zero) then a variation of (6) is received:

$$\frac{\partial(mv_x)}{\partial x} + \frac{\partial(mv_y)}{\partial y} = -\frac{\mathbf{q_v n}}{\rho_V} \tag{6}$$

The m value expressing the depth of the water occurs on the left side of the equation. As in the rest of the equations the h thickness of the water layer is used (that is apparently less than the vertical depth of the flow on sloping surfaces), it is practical to change to h to m in (6).

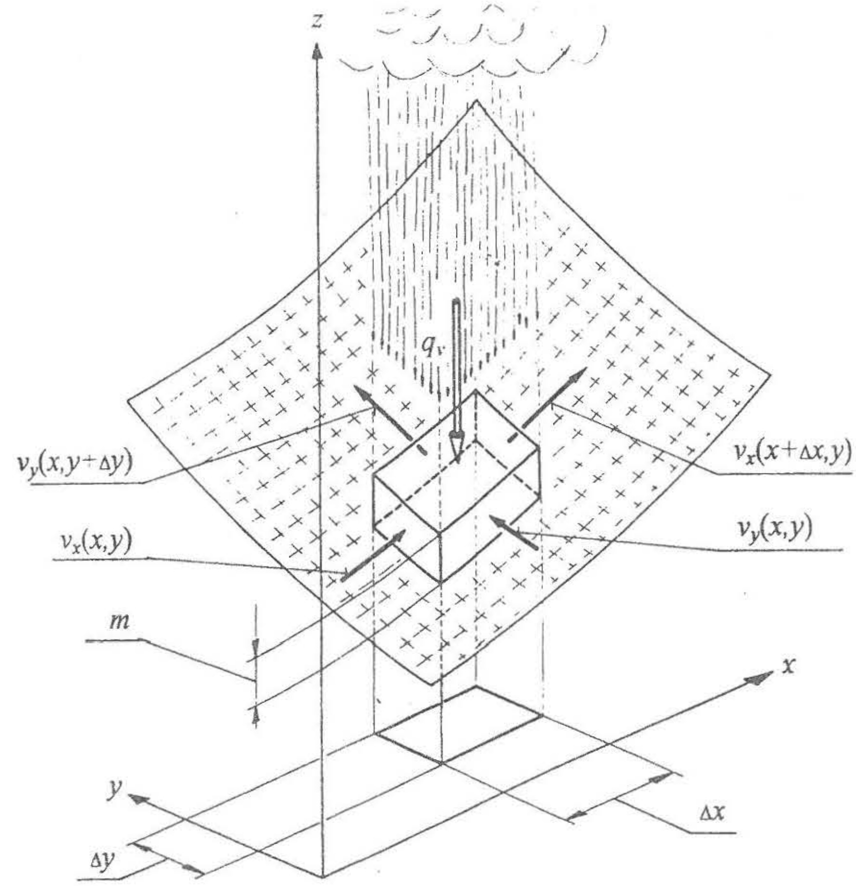

Fig 8: The "infinitely small" element of volume that contains the full depth of the water flowing on the limestone surface, for the application of the mass conservation principle

Due to geometrical considerations the relation between h and m is valid:

$$m = h\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \tag{38}$$

Substituting expression (38) to (6) and executing the assigned derivations after a longish (but elemental) computation the following is received:

$$h\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)+v_x\frac{\partial h}{\partial x}+v_y\frac{\partial h}{\partial y}+v_x h\frac{\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial y\,\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}+v_y h\frac{\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial y^2}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}=$$

$$.=\frac{q_v}{\rho_v}\frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}} \tag{39}$$

(39) connects the differential geometric parameters (gradients in various directions and curvatures) of the rock surface, the flow velocity of the water and the thickness of the flow.

Summary

For the quantitative study of the processes of corrosion of the limestone surface the $v_x(x,y,t)$ and $v_y(x,y,t)$ components of the flow velocity vector, the $h(x,y,t)$ thickness of the liquid film, the $c(x,y,t)$ concentration of dissolved calcium carbonate and the $z=f(x,y,t)$ function determining the limestone surface shall be determined. The listed five unknowns can be derived by the listed five partial differential equations

$$v_x=-\frac{\rho_V g h^2}{3\eta}\frac{\frac{\partial z}{\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}, \tag{40}$$

$$v_y=-\frac{\rho_V g h^2}{3\eta}\frac{\frac{\partial z}{\partial y}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}, \tag{41}$$

$$h\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)+v_x\frac{\partial h}{\partial x}+v_y\frac{\partial h}{\partial y}+v_x h\frac{\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial y\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}+v_y h\frac{\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial y^2}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}=$$

$$=\frac{q_v}{\rho_v}\frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}}, \tag{42}$$

$$\frac{\partial c}{\partial x}v_x+\frac{\partial c}{\partial y}v_y=k\frac{c_e-c}{h}-\frac{c}{\rho_V h}q_v, \tag{43}$$

$$\frac{\partial z}{\partial t}=-k\frac{c_e-c}{\rho_k}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}. \tag{44}$$

The solution of (40)—(44) will be dealt with later.

The Computerized Possibilities for the Solution of the Equations of Karst Corrosion

Unfortunately the general solution for the (40)—(44) equations can not be provided, only particular solutions exist that fit to the initial and boundary conditions (*DUBLJANSZKIJ, J. V. 1989, SZUNYOGH, G. 1995.c*). Apparently some schemes for the solution can be worked out that help in the study of some individual types of tasks.

The(40)—(44) equation system is very befitting for computerized solution because its equations are separable by the unknowns in them thus it is sufficient to solve smaller (three unknowns at most) equation systems. The steps of a numerical solution are described in the followings.

Initial and Boundary Conditions

Be the function reflecting the rock surface at the beginning of the t_o period of study of the karst corrosion:

$$z(x,y,t)=z_0(x,y), \qquad \text{if} \qquad t=t_0 \tag{45}$$

known. The $H_0(x,y,t)$ thickness of the liquid film and the concentration of the calcium carbonate in the solution $C_0(x,y,t)$ at the upper fringes of the sloping rock surface (there where the water arrives to the study site) thus

$$h(x,y,t) = H_0(x,y,t), \qquad \text{if} \quad x,y \in \Gamma, \quad t_0 \le t, \tag{46}$$

and

$$c(x,y,t) = C_0(x,y,t), \qquad \text{if} \quad x,y \in \Gamma, \quad t_0 \le t, \tag{47}$$

shall be known where Γ is the set of the points of the upper limestone surface fringe (*Fig. 9*).

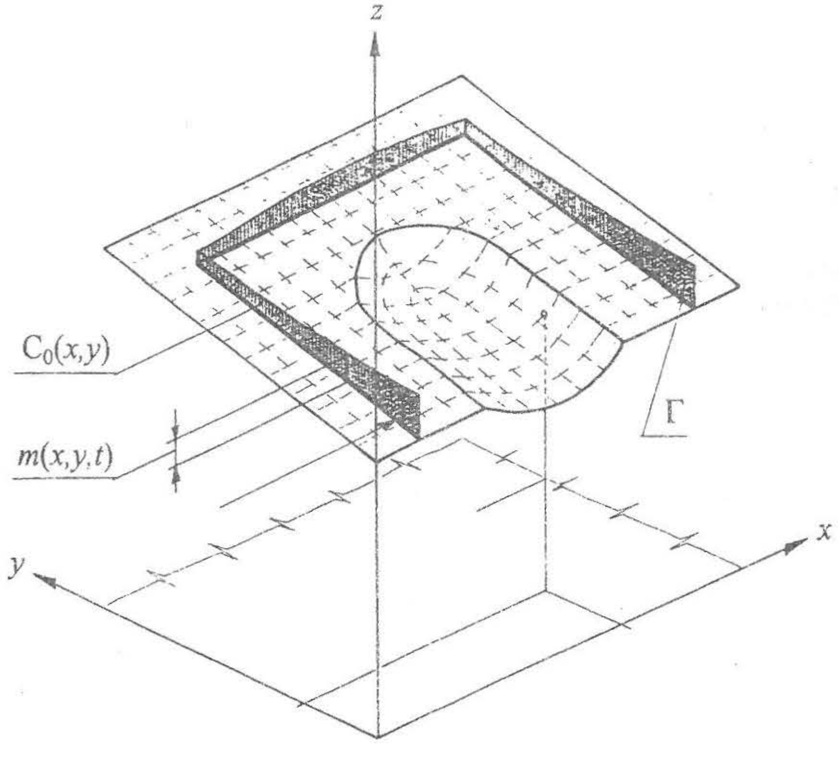

Fig. 9: The initial and boundary conditions of the differential equation system of karst corrosion

The Development of the Karst Corrosion at the Beginning of the Denudation

$$\frac{\partial z_0}{\partial x} = \xi_x^{(0)}(x,y), \qquad \text{if} \qquad t = t_0 \tag{48}$$

$$\frac{\partial z_0}{\partial y} = \xi_y^{(0)}(x,y), \qquad \text{if} \qquad t = t_0, \tag{49}$$

$$\frac{\partial^2 z_0}{\partial x^2} = \zeta_{xx}^{(0)}(x,y), \qquad \text{if} \qquad t = t_0, \tag{50}$$

$$\frac{\partial^2 z_0}{\partial x \partial y} = \zeta_{xy}^{(0)}(x,y), \qquad \text{if} \qquad t = t_0, \tag{51}$$

$$\frac{\partial^2 z_0}{\partial y^2} = \zeta_{yy}^{(0)}(x,y), \qquad \text{if} \qquad t = t_0. \tag{52}$$

The $\xi_x^{(0)}$, $\xi_y^{(0)}$, $\zeta_{xx}^{(0)}$, $\zeta_{xy}^{(0)}$ and $\zeta_{yy}^{(0)}$ functions in (48)-(52) are apparently known. (The $^{(0)}$ index indicates that these derivatives are related to t_0 time). The performance of the derivation (allowing to the computer program) shall not be done by an analytic but a numerical way.

These functions substituted to the equations (40)—(42) a differential equation with three unknowns is gained for the initial v_{x0} és v_{y0} velocity and h_0 thickness of the flow:

$$v_{x0} = -\frac{\rho_V g h_0^2}{3\eta} \frac{\xi_x^{(0)}}{\left(\xi_x^{(0)}\right)^2 + \left(\xi_y^{(0)}\right)^2 + 1}, \tag{53}$$

$$v_{y0} = -\frac{\rho_V g h_0^2}{3\eta} \frac{\xi_y^{(0)}}{\left(\xi_x^{(0)}\right)^2 + \left(\xi_y^{(0)}\right)^2 + 1}, \tag{54}$$

$$h_0\left(\frac{\partial v_{x0}}{\partial x} + \frac{\partial v_{y0}}{\partial y}\right) + v_{x0}\frac{\partial h_0}{\partial x} + v_{y0}\frac{\partial h_0}{\partial y} + v_{x0}h_0 \frac{\xi_x^{(0)}\zeta_{xx}^{(0)} + \xi_y^{(0)}\zeta_{xy}^{(0)}}{\left(\xi_x^{(0)}\right)^2 + \left(\xi_y^{(0)}\right)^2 + 1} + v_{y0}h_0 \frac{\xi_x^{(0)}\zeta_{xy}^{(0)} + \xi_y^{(0)}\zeta_{yy}^{(0)}}{\left(\xi_x^{(0)}\right)^2 + \left(\xi_y^{(0)}\right)^2 + 1} =$$

$$= \frac{q_v}{\rho_v} \frac{1}{\sqrt{\left(\zeta_x^{(0)}\right)^2 + \left(\zeta_y^{(0)}\right)^2 + 1}}. \tag{55}$$

The value of the $H_0(x,y,t)$ function of (46) at $t=t_0$ substituted to (53) and (54) the boundary conditions for velocity, V_{x0} and V_{x0} are gained:

$$V_{x0} = -\frac{\rho_V g H_0^2(x,y,t_0)}{3\eta} \frac{\xi_x^{(0)}}{\left(\xi_x^{(0)}\right)^2 + \left(\xi_y^{(0)}\right)^2 + 1}, \quad \text{if } x,y \in \Gamma \text{ and } t = t_0,$$

(56)

or

$$V_{y0} = -\frac{\rho_V g H_0^2(x,y,t_0)}{3\eta} \frac{\xi_y^{(0)}}{\left(\xi_x^{(0)}\right)^2 + \left(\xi_y^{(0)}\right)^2 + 1}, \quad \text{if } x,y \in \Gamma \text{ and } t = t_0.$$

(57)

With these boundary conditions the (53)—(55) equations can be solved with the method of finite differences (for the $v_{x0}(x,y)$, $v_{y0}(x,y)$ and $h_0(x,y)$ variables) directly or (with the elimination of v_{x0}, v_{y0}) only for $h_0(x,y)$ by the computer. The solution written to (43)

$$\frac{\partial c_0}{\partial x} v_{x0} + \frac{\partial c_0}{\partial y} v_{y0} = k\frac{c_e - c_0}{h_0} - \frac{c_0}{\rho_V h_0} q_v, \tag{58}$$

where the only unknown is the c_0 initial concentration of the solution. Apparently (58) is a partial differential equation and its solution is a two variable function ($c_0 = c_0(x,y)$), but fortunately it is only a linear and simple equation this way its solution on the computer does not raise difficulties. For the solution considering (47) the $c_0(x,y) = C_0(x,y,t)$, if $t = t_0$ boundary condition serves as a supplement.

It can be determined at length what is the velocity of the sinking of a specific (arbitrary) point of x—y coordinates. Equation (44) for $t = t_0$:

$$\left.\frac{\partial z}{\partial t}\right|_{t=0} = -k\frac{c_e - c_0}{\rho_k}\sqrt{\left(\zeta_x^{(0)}\right)^2 + \left(\zeta_y^{(0)}\right)^2 + 1}. \tag{59}$$

The Shape of the Limestone Surface a Short time after the Initial Moment

In the knowledge of the velocity of sinking it can be determined what function describes the shape of the rock surface after the passing of aΔt minute time interval in the next moment ($t_1 = t_0 + \Delta t$).

With the integration of (59):

$$z(x,y,t)\big|_{t=t_1} - z(x,y,t)\big|_{t=t_0} = \int_{t=t_0}^{t_1} -k\frac{c_e - c}{\rho_k}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dt. \tag{60}$$

$z(x,y,t)\big|_{t=t_0}$ means the shape of the rock surface at $t=t_0$ time and $z(x,y,t)\big|_{t=t_1}$ at $t=t_1$ time, that is:

$$z(x,y,t)\big|_{t=t_0} = z_0(x,y), \qquad \text{if} \qquad t=t_0. \tag{61}$$

$$z(x,y,t)\big|_{t=t_1} = z_1(x,y), \qquad \text{if} \qquad t=t_1 \tag{62}$$

The first average value principle of the integration applied at the right side of (60) particularly that it is valid for all continuous and integrable $f(t)$ function in the t_A és t_B interval, that:

$$\int_{t_A}^{t_B} f(t)dt = (t_B - t_A)\cdot f(t^*), \tag{63}$$

where about t^*only that much is known that it is an internal point of the t_A, t_B interval (that is: $t_A \le t^* \le t_B$). t_0 chosen for t_A and t_1 for t_B (60) takes the form:

$$z(x,y,t)\big|_{t=t_1} - z(x,y,t)\big|_{t=t_0} = (t_1 - t_0)\left[-k\frac{c_e - c}{\rho_k}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\right]_{t=t^*} \tag{64}$$

If the difference of t_1 and t_0

$$\Delta t = (t_1 - t_0) \tag{65}$$

is sufficientléy small than t_0, t_1 and t^* are only "slightly" different so writingt^* instead of t_0 into the argument of values at the left of (64) a substantial fault has not been committed. The fault is the smaller as smaller is Δt.

Considering the (48), (49), (61), (62) and (65) signs for the shape of the rock surface in a particular moment will be

$$z_1(x,y) = z_0(x,y) - k\frac{c_e - c_0}{\rho_k}\sqrt{\left(\zeta_x^{(0)}\right)^2 + \left(\zeta_y^{(0)}\right)^2 + 1}\cdot \Delta t \tag{66}$$

$v_{x0}(x,y)$, $v_{y0}(x,y)$, $h_0(x,y)$, $c_0(x,y)$ and $\left.\frac{\partial z}{\partial t}\right|_{t=0}$ expresses all the necessary data to characterize the denudation of the limestone surface at the $t=0$ moment.

The Shape of the Limestone Surface in an Arbitrary Moment

The relations derived in the foregoing retain their validity even when not a t_0 but a t_1 moment is chosen for the initial moment only the indices change, (1) shall be written replacing the "old" (0) indices and (2)-s replace the "old" (1)-s.

$$z_2(x,y) = z_1(x,y) - k\frac{c_e - c_1}{\rho_k}\sqrt{\left(\zeta_x^{(1)}\right)^2 + \left(\zeta_y^{(1)}\right)^2 + 1}\cdot \Delta t. \tag{67}$$

(67) can be computed in the knowledge of (66).

Continuing the same order of ideas the shape of the rock surface can be specified at any arbitrary t_n moment.

1. The derivatives expressing the differential geometric parameters shall be specified in the knowledge of $z_n(x,y,t)$:

$$\xi_x^{(n)}(x,y) = \frac{\partial z_n}{\partial x} \quad \text{if} \quad t = t_n \quad (t_n = t_{n-1} + \Delta t) \tag{68}$$

$$\xi_y^{(n)}(x,y) = \frac{\partial z_n}{\partial y}, \quad \text{if} \quad t = t_n \quad (t_n = t_{n-1} + \Delta t), \tag{69}$$

$$\zeta_{xx}^{(n)}(x,y) = \frac{\partial^2 z_n}{\partial x^2}, \quad \text{if} \quad t = t_n \quad (t_n = t_{n-1} + \Delta t), \tag{70}$$

$$\zeta_{xy}^{(n)}(x,y) = \frac{\partial^2 z_n}{\partial x \partial y}, \quad \text{if} \quad t = t_n \quad (t_n = t_{n-1} + \Delta t), \tag{71}$$

$$\zeta_{yy}^{(n)}(x,y) = \frac{\partial^2 z_n}{\partial y^2}, \quad \text{if} \quad t = t_n \quad (t_n = t_{n-1} + \Delta t), \tag{72}$$

2. Substituting the substitute value of $H_0(x,y,t)$ from (46) at $t=t_0$ to (53) and (54) the boundary conditions for velocity V_{xn} and V_{xn} will be:

$$V_{xn} = -\frac{\rho_V g H_0^2(x,y,t)}{3\eta}\frac{\xi_x^{(n)}}{\left(\xi_x^{(n)}\right)^2 + \left(\xi_y^{(n)}\right)^2 + 1}, \quad \text{if } x,y \in \Gamma, \text{ and } t = t_n, \tag{73}$$

or

$$V_{yn} = -\frac{\rho_V g H_0^2(x,y,t)}{3\eta}\frac{\xi_y^{(n)}}{\left(\xi_x^{(n)}\right)^2 + \left(\xi_y^{(n)}\right)^2 + 1}, \quad \text{if } x,y \in \Gamma, \text{ if } t = t_n. \tag{74}$$

Considering these boundary conditions the (53)—(55) equation system can be solved by computer for the $t=t_n$ moment with the method of finite differences regarding $(v_{xn}(x,y), v_{yn}(x,y)$ and $h_n(x,y)$:

$$v_{xn}(x,y) = -\frac{\rho_V g h_n^2(x,y)}{3\eta} \frac{\xi_x^{(n)}}{\left(\xi_x^{(n)}\right)^2 + \left(\xi_y^{(n)}\right)^2 + 1}, \qquad (75)$$

$$v_{yn}(x,y) = -\frac{\rho_V g h_n^2(x,y)}{3\eta} \frac{\xi_y^{(n)}}{\left(\xi_x^{(n)}\right)^2 + \left(\xi_y^{(n)}\right)^2 + 1}, \qquad (76)$$

$$h_n\left(\frac{\partial v_{xn}}{\partial x} + \frac{\partial v_{yn}}{\partial y}\right) + v_{xn}\frac{\partial h_n}{\partial x} + v_{yn}\frac{\partial h_n}{\partial y} + v_{xn}h_n\frac{\xi_x^{(n)}\zeta_{xx}^{(n)} + \xi_y^{(n)}\zeta_{xy}^{(n)}}{\left(\xi_x^{(n)}\right)^2 + \left(\xi_y^{(n)}\right)^2 + 1} + v_{yn}h_n\frac{\xi_x^{(n)}\zeta_{xy}^{(n)} + \xi_y^{(n)}\zeta_{yy}^{(n)}}{\left(\xi_x^{(n)}\right)^2 + \left(\xi_y^{(n)}\right)^2 + 1} =$$

$$= \frac{q_v}{\rho_v} \frac{1}{\sqrt{\left(\zeta_x^{(n)}\right)^2 + \left(\zeta_y^{(n)}\right)^2 + 1}}. \qquad (77)$$

Writing the solution to (43) a differential equation is received for $c(x,y,t_n)$:

$$\frac{\partial c_n}{\partial x} v_{xn} + \frac{\partial c_n}{\partial y} v_{yn} = k\frac{c_e - c_n}{h_n} - \frac{c_n}{\rho_V h_n} q_v, \qquad (78)$$

That substituted with

$$c_n(x,y) = C_0(x,y,t), \qquad \text{if} \quad x,y \in \Gamma \text{ és} \quad t = t_n \qquad (79)$$

boundary condition it can be solved by computer.

By the ideas followed in (60)

$$z_{n+1}(x,y) = z_n(x,y) - k\frac{c_e - c_n(x,y)}{\rho_k}\sqrt{\left(\zeta_x^{(n)}\right)^2 + \left(\zeta_y^{(n)}\right)^2 + 1} \cdot \Delta t \qquad (80)$$

The procedure of the solution will be (*Fig. 10*):

1. The $z_1(x, y)$ shape of the rock surface is determined at t_1 moment.
2. In the knowledge of $z_1(x, y)$ the shape $z_2(x, y)$ of the rock surface is determined $z_2(x, y)$ for the t_2 moment.
3. This procedure is continued for the series of t_3, t_4, t_5 ... moments as long as the $z_n(x, y)$ function belonging to the t_n moment is achieved. The sought for solution is:

$$z(x,y,t) = z_n(x,y), \quad \text{if} \quad t = t_n. \tag{78}$$

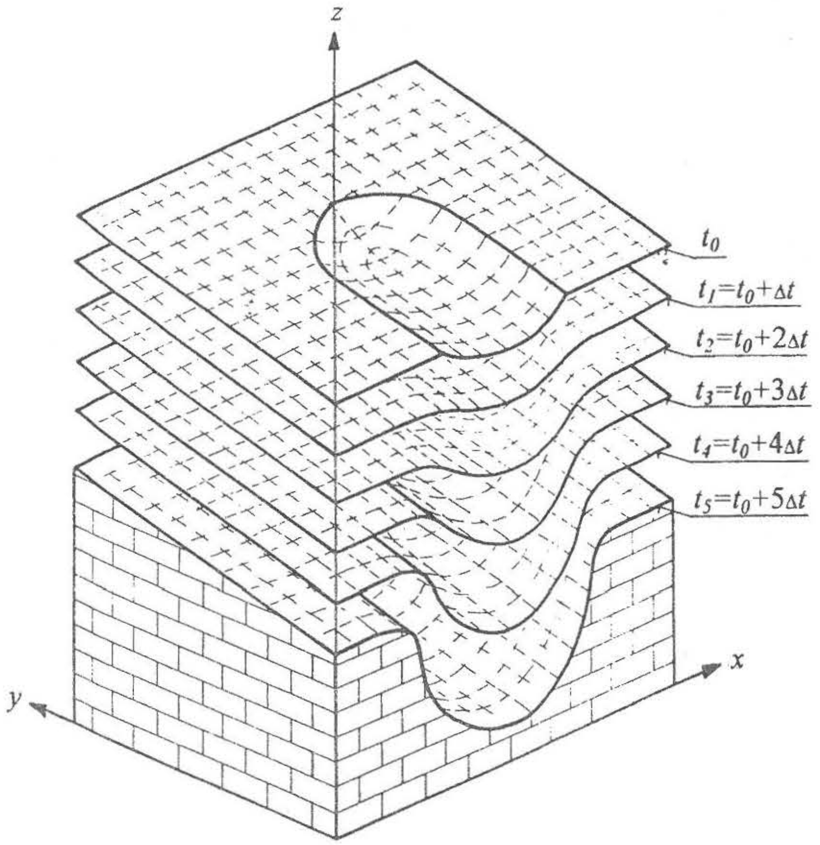

Fig. 10: Basic diagram of the development of a small limestone surface irregularity (e.g. a heel print) to a karren trough for computer modeling

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